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MSE performance of the biased estimators for each individual regression coefficient when relevant regressors are omitted

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In this paper, we consider a linear regression model in which relevant regressors are omitted, and examine the mean squared error (MSE) performance of the Stein-rule (SR), positive-part Stein-rule (PSR), minimum mean squared error (MMSE) and adjusted minimum mean squared error (AMMSE) estimators for each individual regression coefficient. It is shown analytically that the PSR estimator dominates the SR estimator when a condition is satisfied. Also, our numerical results show that the SR estimator dominates the PSR estimator when the condition is not satisfied.

1 Introduction

2 Introduction

Stein (1956) and James and Stein (1961) proposed the Stein-rule estimator (SR estimator) which dominates the ordinary least squares estimator (OLS estimator) in terms of predictive mean squared error (PMSE) if the number of the regression coefficient is large than or equal to three. Furthermore, Branchik (1970) proposed the positive-part Stein-rule estimator (PSR) which dominates the SR estimator.

Theil (1971) proposed the minimum mean squared error estimator (MMSE estimator). However, the MMSE estimator includes unknown parameters. Hence, Farebrother (1975) suggested that unknown parameters are substituted for OLS estimators. In addition, Ohtani (1996) proposed the adjusted minimum mean squared error estimator (AMMSE estimator) which adjusts the degrees of freedom of the MMSE estimator.

Ullah and Ullah (1978) proposed the double k-class estimator (KK estimator) which includes the SR estimator, the MMSE estimator and the AMMSE estimator as special cases. Furthermore, the double k-class estimator has two non-specific parameters. Ohtani (2000) proposed the pre-test double k-class estimator (PTKK estimator) which is the KK estimator conducting a pre-test.

The PTKK estimator includes the SR estimator, the PSR estimator, the MMSE estimator, the AMMSE estimator and the KK estimator as special cases.

Though these estimators dominate the OLS estimator when all the regression coefficients are estimated simultaneously, these estimators do not dominate the OLS estimator when each individual regression coefficient is estimated separately. Ullah (1974) found the exact and approximate moments of the SR estimator for each individual coefficient. Rao and Shinozaki (1978) found a necessary and sufficient condition for the SR estimator for each individual coefficient to dominate the OLS estimator. Ohtani and Kozumi (1996) found the general formulae for the moments of a linear functional of the SR estimator and PSR estimator. Furthermore, Ohtani (1997) derived the exact MSE of the MMSE estimator foe each individual coefficient and showed a sufficient condition for the MMSE estimator for each individual coefficient to dominate the OLS estimator.

These estimators are not unbiased even if the model is specified correctly. In practical situations, the specification of a linear regression model may not be correct. Namely, the misspecification in the linear regression model may be suspected. Mittelhammer (1984) showed that the SR estimator does not dominate the OLS estimator when the model is incorrectly specified. Also, Ohtani (1993) found a sufficient condition for the PSR estimator to dominate the SR estimator in the misspecified model. Besides, Ohtani (1998) shows by numerical analysis that the AMMSE estimator dominates the PSR estimator in the misspecified model. Namba (2002) shows that the PSR estimator dominates the SR estimator in the misspecified model and shows by numerical analysis that the PSR estimator and the AMMSE estimator have much smaller PMSEs than the OLS estimator even when the relevant regressors are omitted.

In this paper we extend the analysis of Ohtani (1997) to the case of the misspecified model. Namely, we consider a linear regression model in which relevant regressors are omitted, and examine the MSE performances of the SR, PSR, MMSE and AMMSE estimators for each individual regression coefficient. In Section 2 the model and the estimators are presented. In Section 3 the formulae for the moment of these estimators in the misspecified model are derived. In Section 4 the MSEs are compared by numerical analysis.

3 Model and the estimator

Consider a linear regression model,

$$Y = X_1\beta_1 + X_2\beta_2 + \epsilon,$$

where Y is an $n \times 1$ vector of observations on a dependent variable, X_1 and X_2 are $n \times k_1$ and $n \times k_2$ matrices of observations on independent variables, β_1 and β_2 are $k_1 \times 1$ and $k_2 \times 1$ vectors of coefficients, and ϵ is an $n \times 1$ vector of disturbances which is normally distributed with $E(\epsilon) = 0$ and $E(\epsilon\epsilon') = \sigma^2 I_n$.

Suppose that the matrix of regressors X_2 is omitted mistakenly and the regression model is specified as

$$Y = X_1\beta_1 + \eta, \tag{1}$$

where $\eta = X_2\beta_2 + \epsilon$. Then, the ordinary least squares (OLS) estimator of β_1 on the misspecified regression model is

$$b_1 = S_1^{-1}X_1'y,$$

where $S_1 = X_1'X_1$. In this case, the double k-class estimator is

$$\hat{\beta}_1 = \left(\frac{b_1'S_1b_1 + \alpha_1 e'e}{b_1'S_1b_1 + \alpha_2 e'e}\right) b_1,$$

where $e = y - X_1b_1$, $\alpha_1 = 1 - k_1 - k_2$ and $\alpha_2 = 1 - k_2$.

Furthermore, the pre-test double k-class estimator is

$$b_{Pkk} = I(F > \tau)\left(\frac{b_1'S_1b_1 + \alpha_1 e'e}{b_1'S_1b_1 + \alpha_2 e'e}\right) b_1,$$

where I(A) is an indicator function such that I(A)=1 if an event A occurs and I(A)=0 otherwise, $F = (b_1'S_1b_1/k_1)/(e'e/(n-k_1))$ is the test statistic for the null hypothesis $H_0 : \beta_1 = 0$, and τ is the critical value of the pretest.

Also, the SR, PSR, MMSE, and AMMSE estimators are written as

$$b_{SR} = (1 - \frac{ae'e}{b_1'S_1b_1})b_1,$$

$$b_{PSR} = max\{0, (1 - \frac{ae'e}{b_1'S_1b_1})\}b_1,$$

$$b_M = (\frac{b_1'S_1b_1}{b_1'S_1b_1 + e'e/(n-k)})b_1,$$

and

$$b_{AM} = (\frac{b_1'S_1b_1/k}{b_1'S_1b_1/k + e'e/(n-k)})b_1.$$

These estimators are included by the PTKK estimator as special cases. The PTKK estimator reduces to the SR estimator when $\tau = 0, \alpha_1 = 2(2-k)/(n-k+2)$ and $\alpha_2 = 0$, and it reduces to the PSR estimator when $\tau = 2(n-k)(k-2)/k(n-k+2), \alpha_1 = 2(2-k)/(n-k+2)$ and $\alpha_2 = 0$. Futhermore it reduces to the MMSE estimator when $\tau = 0, \alpha_1 = 0$ and $\alpha_2 = 1/(n-k)$ and it reduces to the AMMSE estimator when $\tau = 0, \alpha_1 = 0$ and $\alpha_2 = k/(n-k)$.

Furthermore, following Judge and Yancey(1986, p.11.), the model(1) is reparameterized by the following transformation:

$$S_1^{-\frac{1}{2}}S_1S_1^{-\frac{1}{2}} = I_k,$$

$$Z = X_1 S_1^{-\frac{1}{2}}, \qquad X_1 = Z S_1^{\frac{1}{2}}, \qquad \gamma_1 = S_1^{\frac{1}{2}} \beta_1, \qquad \text{and} \quad \beta = S_1^{-\frac{1}{2}} \gamma_1.$$

Hence, the model(1) is written as

$$y = X_1 S_1^{\frac{1}{2}} S_1^{-\frac{1}{2}} \beta_1 + \eta = Z\gamma_1 + \eta.$$

Then, $Z'Z$, c_1 and e are written as

$$Z'Z = S_1^{-\frac{1}{2}} X_1' X_1 S_1^{-\frac{1}{2}} = I_k,$$

$$c_1 = (Z'Z)^{-1} Z'y = Z'y = S_1^{\frac{1}{2}} b_1,$$

$$e = y - Zc_1 = y - Xb_1,$$

and

$$c_1' c_1 = b_1' S b_1.$$

Thus, the double k-class estimator is written as

$$\hat{\gamma}_1 = \left(\frac{c_1'c_1 + \alpha_1 e'e}{c_1'c_1 + \alpha_2 e'e} \right) c_1.$$

If h is the i-th row vector of $S_1^{-\frac{1}{2}}$, the estimator $h'c$ is the i-th element of the SR estimator for β. In addition, the parameter $h'\gamma$ is the i-th element of the parameter for β.

Moreover, F is the test statistics for H_0: $\beta_1 = 0$ which is

$$F = \left(\frac{\frac{c_1'c_1}{k}}{\frac{e'e}{n-k}} \right) > \tau.$$

Thus, the pre-test double k-class estimator for the i-th element of β is written as

$$h'\hat{\gamma}_\tau = \left(I(F > \tau) \frac{c_1'c_1 + \alpha_1 e'e}{c_1'c_1 + \alpha_2 e'e} \right) h'c_1.$$

In the next section, the explicit formula for the MSE of the PTKK estimator is derived.

4 Moment of the estimator

In this section, the explicit formula for the MSE of the PTKK estimator is derived. The bias and MSE of the PTKK estimator are

$$\begin{aligned} \text{Bias}(h'\hat{\gamma}_\tau) &= E[I(F > \tau) h'\hat{\gamma}] - h'\gamma \\ &= E\left[I(F > \tau) \left(\frac{c_1'c_1 + \alpha_1 e'e}{c_1'c_1 + \alpha_2 e'e} \right) (h'c_1) \right] - (h'\gamma), \end{aligned}$$

and

$$
\begin{aligned}
MSE(h'\hat{\gamma}_\tau) &= E[(I(F>\tau)h'\hat{\gamma}-h'\gamma)^2] \\
&= E\left[I(F>\tau)\left(\frac{c_1'c_1+\alpha_1 e'e}{c_1'c_1+\alpha_2 e'e}\right)^2 (h'c_1)^2\right] \\
&\quad -2h'\gamma E\left[I(F>\tau)\left(\frac{c_1'c_1+\alpha_1 e'e}{c_1'c_1+\alpha_2 e'e}\right) h'c_1\right] + (h'\gamma)^2.
\end{aligned}
$$

If we define the functions as

$$H(p:q:\alpha_1:\alpha_2:\tau) = E\left[I(F>\tau)\left(\frac{c_1'c_1+\alpha_1 e'e}{c_1'c_1+\alpha_2 e'e}\right)^p (h'c_1)^{2q}\right],$$

and

$$J(p:q:\alpha_1:\alpha_2:\tau) = E\left[I(F>\tau)\left(\frac{c_1'c_1+\alpha_1 e'e}{c_1'c_1+\alpha_2 e'e}\right)^p (h'c_1)^{2q+1}\right],$$

then the bias and the MSE of the PTKK estimator are written as

$$\text{Bias}(h'\hat{\gamma}_\tau) = J(1:0:\alpha_1:\alpha_2:\tau) - (h'\gamma),$$

and

$$MSE(h'\hat{\gamma}_\tau) = H(2:1:\alpha_1:\alpha_2:\tau) - 2h'\gamma J(1:0:\alpha_1:\alpha_2:\tau) + (h'\gamma)^2.$$

As is shown in the appendix A, the explicit formulae for $H(p:q:\alpha_1:\alpha_2:\tau)$ and $J(p:q:\alpha_1:\alpha_2:\tau)$ are

$$
\begin{aligned}
H(\quad p &: q:\alpha_1:\alpha_2:\tau) \\
&= (2\sigma^2)^q \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{m=0}^{\infty} w_i(\lambda_1)w_j(\lambda_2)w_m(\lambda_3)G_{ij}(p,q;\alpha_1,\alpha_2;\tau),
\end{aligned}
$$

and

$$
\begin{aligned}
J(\quad p &: q:\alpha_1:\alpha_2:\tau) = h'(\gamma+Z'X_2\beta_2)(2\sigma^2)^q \\
&\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{m=0}^{\infty} w_i(\lambda_1)w_j(\lambda_2)w_m(\lambda_3)\times G_{i+1j}(p,q;\alpha_1,\alpha_2;\tau),
\end{aligned}
$$

where $w_i(\lambda) = exp(-\lambda/2)(\lambda/2)^i/i!$,

$$\lambda_1 = \frac{(h'\gamma+h'Z'X_2\beta_2)^2}{\sigma^2},$$

$$\lambda_2 = \frac{(\gamma+Z'X_2\beta_2)'(I_k-hh')(\gamma+Z'X_2\beta_2)}{\sigma^2},$$

$$\lambda_3 = \frac{(X_2\beta_2)'(I_n-ZZ')(X_2\beta_2)}{\sigma^2},$$

and

$$G_{ij}(\quad p \quad, q; \alpha_1, \alpha_2; \tau) = \frac{\Gamma(\frac{1}{2}+q+i)\Gamma(\frac{n}{2}+q+i+j+m)}{\Gamma(\frac{1}{2}+i)\Gamma(\frac{n-k}{2}+m)\Gamma(\frac{k}{2}+q+i+j)}$$
$$\times \int_{\bar{\tau}}^{1} \left(\frac{\alpha_1+(1-\alpha_1)t}{\alpha_2+(1-\alpha_2)t}\right)^p t^{\frac{k}{2}+q+i+j-1}(1-t)^{\frac{n-k}{2}+m-1}dt,$$

where $\bar{\tau} = k\tau/(k\tau+n-k)$.

As is shown in the appendix A,

$$\frac{\text{MSE}(h'\hat{\gamma}_p)}{\partial\tau}$$
$$= \left(\frac{\alpha_1(n-k)+k\tau}{\alpha_2(n-k)+k\tau}\right)\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{m=0}^{\infty} w_i(\lambda_1)w_j(\lambda_2)w_m(\lambda_3)$$
$$\times\frac{\Gamma(\frac{3}{2}+i)\Gamma(\frac{n}{2}+i+j+m+1)}{}\Gamma(\frac{1}{2}+i)\Gamma(\frac{n-k}{2}+m)\Gamma(\frac{k}{2}+i+j+1)$$
$$\times\frac{k^{k/2+i+j+1}(n-k)^{(n-k)/2+m}\tau^{k/2+i+j}}{(k\tau+n-k)^{n/2+i+j+m+1}}$$
$$\times\left[-2\sigma^2\left(\frac{\alpha_1(n-k)+k\tau}{\alpha_2(n-k)+k\tau}\right)+2\frac{1}{\frac{1}{2}+i}h'\gamma(h'\gamma+h'Z'X_2\beta_2)\right]. \tag{2}$$

From(2), the conditions for $\text{MSE}(h'\hat{\gamma}_p)$ to monotonically decrease are

$$h'\gamma(h'\gamma+h'Z'X_2\beta_2)>0, \tag{3}$$

and

$$\min(-\alpha_1,-\alpha_2)<\tau k/(n-k)<\max(-\alpha_1,-\alpha_2). \tag{4}$$

Without the specification error, the condition (3) is satisfied. Because $h'\hat{\gamma}_p$ reduces to the SR estimator when $\tau=0, \alpha_1=2(2-k)/(n-k+2)$ and $\alpha_2=0$, and it reduces to the PSR estimator when $\tau=2(n-k)(k-2)/k(n-k+2), \alpha_1=2(2-k)/(n-k+2)$ and $\alpha_2=0$, the pre-test SR estimator satisfies the condition (4), and the PSR estimator dominates the SR estimator with the condition (3). Furthermore, the MMSE estimator and the AMMSE estimator do not satisfy the condition (4).

When all the regression coefficients are estimated simultaneously, and when the linear regression model has the specification error, Namba (2002) showed that the PSR estimator dominates the SR estimator.

Because further theoretical analysis of the MSEs of the PTKK estimator is difficult, they are compared numerically in the next section.

5 Numerical Analysis

In this section, the MSE performances of the SR, PSR, MMSE and AMMSE estimators are compared by numerical evaluations. The parameter values used in the numerical evaluations are $k_1 = 3$, $h'\gamma(h'\gamma + h'Z'X_2\beta_2) = 8$, 4, 0, -4, -8, $\lambda_1 + \lambda_2 + \lambda_3 = 5$, 10, 15, and various values for $\lambda_1 + \lambda_2 =$ and λ_1. To compare the MSEs of the estimators, we evaluate the values of relative MSE defined as $MSE(h'\bar{\gamma})/MSE(h'c)$, where $h'\bar{\gamma}$ is any estimator of $h'\gamma$ and c is the OLS estimator. Thus, the estimator $h'\bar{\gamma}$ has smaller MSE than the OLS estimator when the value of relative MSE is smaller than unity.

Tables 1 and 2 show the relative MSE of the SR, PSR, MMSE and AMMSE estimators for $h'\gamma(h'\gamma+h'Z'X_2\beta_2) = 8$ and 4. We see from Tables 1 and 2 that the OLS estimator has smaller MSE than the SR, PSR, MMSE and AMMSE estimators over the wide region of the parameter space considered here. This confirms the precedent results that the SR, PSR, MMSE and AMMSE estimators no longer dominate the OLS estimator when there are omitted variables or when each individual regression coefficient is estimated separately.

Table 1 and 2 are obtained under the condition that the value of $h'\gamma(h'\gamma + h'Z'X_2\beta_2)$ is positive. This is the condition for the PSR estimator to dominate the SR estimator. The bias of the PTKK estimator is written as

$$\text{Bias}(h'\hat{\gamma}) = (h'\gamma + Z'X_2\beta_2)S(\alpha_1 : \alpha_2 : \tau) - h'\gamma,$$

where

$$S(\alpha_1\ \alpha_2 : \tau) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{m=0}^{\infty} w_i(\lambda_1)w_j(\lambda_2)w_m(\lambda_3)G_{i+1j}(1,0;\alpha_1,\alpha_2;\tau) > 0.$$

Thus, when $h'\gamma(h'\gamma + h'Z'X_2\beta_2) > 0$, the sign of the bias is indefinite. Also, although the PSR estimator dominates the SR estimator, the biased estimators have larger MSE than the OLS estimator over the wide region of the parameter space. When there is no specification error, the bias of the biased estimator is usually negative if a parameter to be estimated is positive and vice versa, and the biased estimators have smaller MSE than the OLS estimator over the wide region of the parameter space. When $h'\gamma(h'\gamma + h'Z'X_2\beta_2) > 0$, the above unexpected results that the biased estimators have larger MSE than the OLS estimator seem to be caused by the term $h'Z'X_2\beta_2$ due to the specification error.

We also see from Tables 1 and 2 that when the value of $\lambda_1 + \lambda_2 + \lambda_3$ is large and the value of $\lambda_1+\lambda_2$ is close to the value of $\lambda_1+\lambda_2+\lambda_3$ (e.g., $\lambda_1+\lambda_2+\lambda_3 = 15$ and $\lambda_1 + \lambda_2 = 9.6$ in Table 1), the relative MSE gets smaller than unity. The parameter λ_3 depends on β_2 though it does not depend on β_1. This indicates that as the magnitude of specification error gets large, the value of λ_3 gets large. This indicates that when the value of $\lambda_1+\lambda_2$ is close to the value of $\lambda_1+\lambda_2+\lambda_3$, the value of λ_3 is close to zero. And thus the magnitude of specification error is small. Our result shows that when the magnitude of specification error is small, the SR, PSR, MMSE and AMMSE estimators have smaller MSE than the OLS estimator.

Comparing the SR and PSR estimators, the PSR estimator has smaller MSE than the SR estimator. Since the results shown in Tables 1 and 2 are obtained under the condition (3) (i.e., $h'\gamma(h'\gamma + h'Z'X_2\beta_2) = 8$ and 4), this is a natural result. Also, comparing the MMSE and AMMSE estimators, the MMSE estimator has smaller MSE than the AMMSE estimator over the wide region of the parameter space when $h'\gamma(h'\gamma + h'Z'X_2\beta_2) = 8$ and 4.

Table 3 shows the MSEs of the estimators when $h'\gamma(h'\gamma + h'Z'X_2\beta_2) = 0$. We see from Table 3 that when $h'\gamma(h'\gamma+h'Z'X_2\beta_2) = 0$, the PSR estimator has smaller MSE than the SR estimator. Also, the SR, PSR, MMSE and AMMSE estimators have smaller MSE than the OLS estimator over the wide region of the parameter space, and the AMMSE estimator has smallest MSE among the SR, PSR, MMSE and AMMSE estimators.

Table 3 is obtained under the condition that the value of $h'\gamma(h'\gamma+h'Z'X_2\beta_2)$ is zero. When this condition is satisfied, we have that $h'\gamma = 0$ or $h'\gamma + h'Z'X_2\beta_2 = 0$. When $h'\gamma+h'Z'X_2\beta_2 = 0$, the sign of the bias is negative($\text{Bias}(h'\hat{\gamma}) = -h'\gamma$). However, when $h'\gamma = 0$, the sign of the bias depends on the value of $h'\gamma+h'Z'X_2\beta_2$. However, as is shown above, the biased estimators have smaller MSE than the OLS estimator over the wide region of the parameter space. This indicates that the effect of the specification error on the sampling properties of the biased estimators is small relative to the case of $h'\gamma(h'\gamma + h'Z'X_2\beta_2) > 0$.

We see from Tables 4 and 5 that when the condition that the value of $h'\gamma(h'\gamma + h'Z'X_2\beta_2)$ is not satisfied, the PSR estimator has larger MSE than the SR estimator. Also, comparing the MMSE and AMMSE estimators, the AMMSE estimator has smaller MSE than the MMSE estimator. We also see that although the SR, PSR, MMSE and AMMSE estimators have larger MSE than the OLS estimator when $h'\gamma(h'\gamma+h'Z'X_2\beta_2) > 0$, these four shrinkage estimators have smaller MSE than the OLS estimator when $h'\gamma(h'\gamma+h'Z'X_2\beta_2) < 0$. This indicates that when the condition that the PSR estimator dominates the SR estimator is not satisfied, these four shrinkage estimators can have smaller MSE than the OLS estimator.

Tables 4 and 5 are obtained under the condition that the value of $h'\gamma(h'\gamma + h'Z'X_2\beta_2)$ is negative. In this case, the condition for the PSR estimator to dominate the SR estimator is not satisfied. When $h'\gamma(h'\gamma + h'Z'X_2\beta_2)$ is negative and $h'\gamma$ is positive, the bias is negative. Conversely, when $h'\gamma(h'\gamma + h'Z'X_2\beta_2)$ is negative and $h'\gamma$ is negative, the bias is positive. This indicates that when the condition for the PSR estimator to dominate the SR estimator is not satisfied, the sign of the bias depends on the sign of $h'\gamma$. (Note that $h'\gamma$ is a regression coefficient to be estimated.) As is shown above, the biased estimators have smaller MSE than the OLS estimator over the wide region of the parameter space. Our numerical results show that when $h'\gamma(h'\gamma + h'Z'X_2\beta_2)$ is negative, the effect of the specification error on the sampling properties of the biased estimators is small relative to the case of $h'\gamma(h'\gamma + h'Z'X_2\beta_2) > 0$, though the condition for the PSR estimator to dominate SR estimator is not satisfied.

6 Concluding remarks

Namba (2002) showed that the PSR estimator dominates the SR estimator in a misspecified model when all the regression coefficients are estimated simultaneously. When each individual regression coefficient is separately estimated, without the condition (3) it is not known whether the PSR estimator dominates the SR estimator. When the condition (3) is not satisfied, our numerical results show that the PSR estimator does not dominates the SR estimator but the SR estimator dominates the PSR estimator.

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Appendix A

In this Appendix A, the formulae for $H(p:q:\alpha_1:\alpha_2:\tau)$ and $J(p:q:\alpha_1:\alpha_2:\tau)$ are derived. First, the formulae for $H(p:q:\alpha_1:\alpha_2:\tau)$ is derived. If we define u_1,u_2 and u_3 as

$$u_1 = \frac{(h'c)^2}{\sigma^2} \sim \chi_1^2(\lambda_1),$$

$$u_2 = \frac{c'[I_k - hh']c}{\sigma^2} \sim \chi_{k-1}^2(\lambda_2),$$

and

$$u_3 = \frac{e'e}{\sigma^2} \sim \chi_{n-k}^2,$$

where

$$\lambda_1 = \frac{(h'\gamma_1 + h'Z'X_2\beta_2)^2}{\sigma^2},$$

$$\lambda_2 = \frac{(\gamma_1 + Z'X_2\beta_2)'(I_k - hh')(\gamma_1 + Z'X_2\beta_2)}{\sigma^2},$$

and

$$\lambda_3 = \frac{(X_2\beta_2)'(I_n - ZZ')(X_2\beta_2)}{\sigma^2},$$

then u_1,u_2 and u_3 are ditributed as the noncentral chi-square distribution. Further,u_1,u_2 and u_3 are mutually independent. Using u_1,u_2 and u_3, $H(p:q:\alpha_1:\alpha_2:\tau)$ can be expressed as

$$\begin{aligned} H(\quad p \quad & : q:\alpha_1:\alpha_2:\tau) \\ &= (\sigma^2)^q \int\int\int_R \sum_{i=0}^{\infty}\sum_{i=0}^{\infty}\sum_{i=0}^{\infty} K_{ijm} \left(\frac{u_1+u_2+\alpha_1 u_3}{u_1+u_2+\alpha_2 u_3}\right)^p \\ & \quad \times u_1^{\frac{1}{2}+q+i-1} u_2^{\frac{k-1}{2}+j-1} u_3^{\frac{n-k}{2}+m-1} e^{-\frac{u_i+u_2+u_3}{2}} du_1\, du_2\, du_3, \end{aligned} \tag{5}$$

where

$$K_{ijm} = \frac{w_i(\lambda_1)w_j(\lambda_2)w_m(\lambda_3)}{2^{\frac{n}{2}+i+j+m}\Gamma(1/2+i)\Gamma((k-1)/2+j)\Gamma((n-k)/2+m)},$$

$w_i(\lambda) = exp(-\lambda/2)(\lambda/2)^i/i!$, and R is the region such that $(u_1+u_2)/u_3 > k\tau/(n-k) = \tau^*$.

Let $v_1 = (u_1+u_2)/u_3$,$v_2 = u_1u_3/(u_1+u_2)$ and $v_3 = u_3$. Using,v_1,v_2 and v_3 in (5), We obtain

$$\begin{aligned} (5) &= (\sigma^2)^q \sum_{i=0}^{\infty}\sum_{i=0}^{\infty}\sum_{i=0}^{\infty} K_{ijm} \int_{\tau^*}^{\infty}\int_0^{v_3}\int_0^{\infty} \left(\frac{\alpha_1+v_1}{\alpha_2+v_1}\right)^p \\ & \quad \times v_1^{\frac{k}{2}+q+i+j-1} v_2^{\frac{1}{2}+q+i-1} v_3^{\frac{n-k}{2}+m} (v_3-v_2)^{\frac{k-1}{2}+j-1} \\ & \quad \times e^{-\frac{(1+v_1)v_3}{2}} dv_1\, dv_2\, dv_3. \end{aligned} \tag{6}$$

Again, substituting $z_1 = v_2/v_3$, (6) reduces to

$$\begin{aligned} (5) &= (\sigma^2)^q \sum_{i=0}^{\infty}\sum_{i=0}^{\infty}\sum_{i=0}^{\infty} K_{ijm} \frac{\Gamma(\frac{1}{2}+q+i)\Gamma(\frac{k-1}{2}+j)}{\Gamma(\frac{k}{2}+q+i+j)} \int_{\tau^*}^{\infty}\int_0^{\infty} \left(\frac{\alpha_1+v_1}{\alpha_2+v_1}\right)^p \\ & \quad \times v_1^{\frac{k}{2}+q+i+j-1} v_3^{\frac{n}{2}+q+i+j+m-1} e^{-\frac{(1+v_1)v_3}{2}} dv_1\, dv_3. \end{aligned} \tag{7}$$

Futher,substituting $z_2 = (1+v_1)v_3/2$, (7) reduces to

$$\begin{aligned} (5) &= (\sigma^2)^q \sum_{i=0}^{\infty}\sum_{i=0}^{\infty}\sum_{i=0}^{\infty} K_{ijm} 2^{\frac{n}{2}+q+i+j+m} \\ & \quad \times \frac{\Gamma(\frac{1}{2}+q+i)\Gamma(\frac{k-1}{2}+j)\Gamma(\frac{n}{2}+q+i+j+m)}{\Gamma(\frac{k}{2}+q+i+j)} \\ & \quad \times \int_{\tau^*}^{\infty} \left(\frac{\alpha_1+v_1}{\alpha_2+v_1}\right)^p \frac{v_1^{\frac{k}{2}+q+i+j-1}}{(1+v_1)^{\frac{n}{2}+q+i+j+m}} dv_1. \end{aligned} \tag{8}$$

Finally, making use of the change of variable $t = v_1/(1+v_1)$, we obtain

$$\begin{aligned}(5) \quad = \quad & (\sigma^2)^q \sum_{i=0}^{\infty}\sum_{i=0}^{\infty}\sum_{i=0}^{\infty} K_{ijm} 2^{\frac{n}{2}+q+i+j+m} \\ & \times \frac{\Gamma(\frac{1}{2}+q+i)\Gamma(\frac{k-1}{2}+j)\Gamma(\frac{n}{2}+q+i+j+m)}{\Gamma(\frac{k}{2}+q+i+j)} \\ & \times \int_{\bar{\tau}}^{1} \left(\frac{\alpha_1+(1-\alpha_1)t}{\alpha_2+(1-\alpha_2)t}\right)^p t^{\frac{k}{2}+q+i+j-1}(1-t)^{\frac{n-k}{2}+m-1}dt,\end{aligned}$$

where $\bar{\tau} = k\tau/(k\tau+n-k)$.Namely

$$(5) = (2\sigma^2)^q \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{m=0}^{\infty} w_i(\lambda_1)w_j(\lambda_2)w_m(\lambda_3)G_{ij}(p,q;\alpha_1,\alpha_2;\tau), \qquad (9)$$

where

$$\begin{aligned}G_{ij}(\quad p \quad & ,q;\alpha_1,\alpha_2;\tau) \\ = \quad & \frac{\Gamma(\frac{1}{2}+q+i)\Gamma(\frac{n}{2}+q+i+j+m)}{\Gamma(\frac{1}{2}+i)\Gamma(\frac{n-k}{2}+m)\Gamma(\frac{k}{2}+q+i+j)} \\ & \times \int_{\bar{\tau}}^{1} \left(\frac{\alpha_1+(1-\alpha_1)t}{\alpha_2+(1-\alpha_2)t}\right)^p t^{\frac{k}{2}+q+i+j-1}(1-t)^{\frac{n-k}{2}+m-1}dt\end{aligned}$$

Differentiating (9) with respect to γ, we have

$$\begin{aligned}& \frac{\partial H(p:q:\alpha_1:\alpha_2:\tau)}{\partial\gamma} \\ & = (2\sigma^2)^q \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{m=0}^{\infty} \left[\frac{\partial w_i(\lambda_1)}{\partial\gamma}w_j(\lambda_2)w_m(\lambda_3) + \frac{\partial w_j(\lambda_2)}{\partial\gamma}w_i(\lambda_1)w_m(\lambda_3)\right] G_{ij} \\ & = -\frac{hh'(\gamma+Z'X_2\beta)}{\sigma^2}(2\sigma^2)^q w_i(\lambda_1)w_j(\lambda_2)w_m(\lambda_3)G_{ij}(p,q;\alpha_1,\alpha_2;\tau) \\ & \quad +\frac{hh'(\gamma+Z'X_2\beta)}{\sigma^2}(2\sigma^2)^q w_i(\lambda_1)w_j(\lambda_2)w_m(\lambda_3)G_{i+1j}(p,q;\alpha_1,\alpha_2;\tau) \\ & \quad -\frac{(I_k-hh')(\gamma+Z'X_2\beta)}{\sigma^2}(2\sigma^2)^q w_i(\lambda_1)w_j(\lambda_2)w_m(\lambda_3)G_{ij}(p,q;\alpha_1,\alpha_2;\tau) \\ & \quad +\frac{(I_k-hh')(\gamma+Z'X_2\beta)}{\sigma^2}(2\sigma^2)^q w_i(\lambda_1)w_j(\lambda_2)w_m(\lambda_3)G_{i+1j}(p,q;\alpha_1,\alpha_2;\tau),\end{aligned}$$

where we difne $w_{-1}(\lambda_1) = w_{-1}(\lambda_2) = 0$. Since $h'h = 1$ and $h'(I_k - hh') = 0$, we obtain

$$\begin{aligned}h' \quad & \frac{\partial H(p:q:\alpha_1:\alpha_2:\tau)}{\partial\gamma} \\ = \quad & -\frac{h'(\gamma+Z'X_2\beta)}{\sigma^2}H(p:q:\alpha_1:\alpha_2:i) \\ & +\frac{h'(\gamma+Z'X_2\beta)}{\sigma^2}(2\sigma^2)^q w_i(\lambda_1)w_j(\lambda_2)w_m(\lambda_3)G_{i+1j}(p,q;\alpha_1,\alpha_2;\tau).\end{aligned}$$

Expressing (9), by c and $e'e$, we have

$$H(p : q : \alpha_1 : \alpha_2 : \tau) = \int\int_{F>\tau} \left(\frac{c_1'c_1 + \alpha_1 e'e}{c_1'c_1 + \alpha_2 e'e}\right)^p (h'c_1)^{2q} f_c(c) f_{e'e}(e'e) dc\, de'e. \tag{10}$$

Differentiating (10) with respect to γ, we have

$$\begin{aligned}
&\frac{\partial H(p : q : \alpha_1 : \alpha_2 : \tau)}{\partial \gamma} \\
&= \frac{1}{\sigma^2}\int\int_{F>\tau} \left(\frac{c_1'c_1 + \alpha_1 e'e}{c_1'c_1 + \alpha_2 e'e}\right)^p (h'c_1)^{2q} c f_c(c) f_{e'e}(e'e) \\
&\quad - \frac{\gamma + Z'X_2\beta}{\sigma^2}\int\int_{F>\tau} \left(\frac{c_1'c_1 + \alpha_1 e'e}{c_1'c_1 + \alpha_2 e'e}\right)^p (h'c_1)^{2q} c f_c(c) f_{e'e}(e'e) \\
&= \frac{1}{\sigma^2} E\left[I(F>\tau)\left(\frac{c_1'c_1 + \alpha_1 e'e}{c_1'c_1 + \alpha_2 e'e}\right)^p (h'c_1)^{2q} c\right] \\
&\quad - \frac{\gamma + Z'X_2\beta}{\sigma^2} H(p : q : \alpha_1 : \alpha_2 : \tau).
\end{aligned}$$

Multiplying $h'$from the left, we obtain

$$\begin{aligned}
h'\frac{\partial H(p : q : \alpha_1 : \alpha_2 : \tau)}{\partial \gamma} &= \frac{1}{\sigma^2} E\left[I(F>\tau)\left(\frac{c_1'c_1 + \alpha_1 e'e}{c_1'c_1 + \alpha_2 e'e}\right)^p (h'c_1)^{2q+1}\right] \\
&- \frac{h'(\gamma + Z'X_2\beta)}{\sigma^2} H(p : q : \alpha_1 : \alpha_2 : \tau).
\end{aligned} \tag{11}$$

Equating (10) and (11), we obtain

$$\begin{aligned}
J(p : q : \alpha_1 : \alpha_2 : \tau) &= h'(\gamma + Z'X_2\beta)(2\sigma^2)^q \\
&\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{m=0}^{\infty} w_i(\lambda_1) w_j(\lambda_2) w_m(\lambda_3) G_{i+1j}(p, q; \alpha_1, \alpha_2; \tau).
\end{aligned}$$

Finally, the formulae for $J(p : q : \alpha_1 : \alpha_2 : \tau)$ is derived.

Appendix B

In this Appendix B,MSE($h'\hat{\gamma}_p$) is differentiated with respect to τ.

First $G_{ij}(p, q; \alpha_1, \alpha_2; \tau)$ is differentiated with respect to τ.

$$\begin{aligned}
&\frac{\partial G_{ij}(p,q;\alpha_1,\alpha_2;\tau)}{\partial \tau} \\
&= -\frac{\Gamma(\frac{1}{2} + q + i)\Gamma(\frac{n}{2} + q + i + j + m)}{\Gamma(\frac{1}{2} + i)\Gamma(\frac{n-k}{2} + m)\Gamma(\frac{k}{2} + q + i + j)} \\
&\quad \times \left(\frac{\alpha_1(n-k) + k\tau}{\alpha_2(n-k) + k\tau}\right)^p \frac{k^{k/2+q+i+j}(n-k)^{(n-k)/2+m}\tau^{k/2+q+i+j-1}}{(k\tau + n - k)^{n/2+q+i+j+m}}.
\end{aligned} \tag{12}$$

From (12) ,MSE($h'\hat{\gamma}_p$) is differentiated with respect to τ.

$$
\begin{aligned}
\frac{\text{MSE}(h'\hat{\gamma}_p)}{\partial\tau} &= \left(\frac{\alpha_1(n-k)+k\tau}{\alpha_2(n-k)+k\tau}\right)\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{m=0}^{\infty} w_i(\lambda_1)w_j(\lambda_2)w_m(\lambda_3) \\
&\quad\times\frac{\Gamma(\frac{3}{2}+i)\Gamma(\frac{n}{2}+i+j+m+1)}{\Gamma(\frac{1}{2}+i)\Gamma(\frac{n-k}{2}+m)\Gamma(\frac{k}{2}+i+j+1)} \\
&\quad\times\frac{k^{k/2+i+j+1}(n-k)^{(n-k)/2+m}\tau^{k/2+i+j}}{(k\tau+n-k)^{n/2+i+j+m+1}} \\
&\quad\times\left[-2\sigma^2\left(\frac{\alpha_1(n-k)+k\tau}{\alpha_2(n-k)+k\tau}\right)+2\frac{1}{\frac{1}{2}+i}h'\gamma(h'\gamma+h'Z'X_2\beta_2)\right].
\end{aligned}
$$

Table 1: $h'\gamma_1(h'\gamma + h'Z'X_2\beta_2) = 8$,n=10,k=3

$\lambda_1+\lambda_2+\lambda_3$	$\lambda_1+\lambda_2$	λ_1	SR	PSR	MMSE	AMMSE
5	1	0.2	1.017304	1.011698	1.005633	1.010581
		0.4	1.035114	1.023585	1.011313	1.021268
		0.6	1.053429	1.035651	1.017032	1.032049
		0.8	1.072249	1.047884	1.022782	1.042907
	2	0.4	1.025792	1.019406	1.009399	1.018613
		0.8	1.052816	1.039331	1.0189	1.037497
		1.2	1.081021	1.05968	1.028437	1.056532
		1.6	1.110326	1.08033	1.037927	1.075568
	3	0.6	1.029026	1.023757	1.011674	1.024309
		1.2	1.05964	1.04823	1.023422	1.048924
		1.8	1.091569	1.073076	1.035018	1.073405
		2.4	1.124399	1.097827	1.046171	1.097181
	4	0.8	1.029113	1.025418	1.01276	1.027876
		1.6	1.059658	1.051465	1.025442	1.05583
		2.4	1.090879	1.077314	1.037522	1.08279
		3.2	1.121646	1.101805	1.04831	1.107316
10	2	0.4	1.040629	1.026276	1.012511	1.022957
		0.8	1.083721	1.053327	1.025169	1.046301
		1.2	1.129273	1.081034	1.037887	1.069886
		1.6	1.177232	1.109244	1.05056	1.093536
	4	0.8	1.049467	1.038821	1.01859	1.037372
		1.6	1.102713	1.078979	1.037116	1.075085
		2.4	1.158816	1.119309	1.054831	1.111764
		3.2	1.216225	1.158146	1.070745	1.145544
	6	1.2	1.04662	1.04121	1.020353	1.044409
		2.4	1.095146	1.08253	1.03961	1.08741
		3.6	1.141657	1.119905	1.055341	1.124064
		4.8	1.180243	1.147634	1.064379	1.147764
	8	1.6	1.03938	1.037235	1.019248	1.045205
		3.2	1.0767	1.071506	1.035621	1.085147
		4.8	1.103429	1.094317	1.044158	1.108804
		6.4	1.109064	1.095588	1.039452	1.103694
15	3	0.6	1.066906	1.042511	1.020014	1.036273
		1.2	1.139953	1.086709	1.040223	1.073263
		1.8	1.219027	1.132062	1.06025	1.110365
		2.4	1.303712	1.177823	1.079614	1.146784
	6	1.2	1.070325	1.057757	1.027335	1.055922
		2.4	1.146066	1.116568	1.053345	1.110647
		3.6	1.222254	1.17109	1.074821	1.158172
		4.8	1.290744	1.213602	1.087557	1.190352
	9	1.8	1.059021	1.054856	1.027229	1.061762
		3.6	1.11423	1.103902	1.048881	1.113972
		5.4	1.148942	1.130663	1.055633	1.136804
		7.2	1.144519	1.117848	1.038938	1.110827
	12	2.4	1.044212	1.043242	1.022886	1.05667
		4.8	1.072648	1.070164	1.034473	1.089702
		7.2	1.059296	1.055057	1.020357	1.064724
		9.6	0.998364	0.992855	0.979651	0.976619

Table 2: $h'\gamma_1(h'\gamma + h'Z'X_2\beta_2) = 4$,n=10,k=3

$\lambda_1+\lambda_2+\lambda_3$	$\lambda_1+\lambda_2$	λ_1	SR	PSR	MMSE	AMMSE
5	1	0.2	1.035393	1.021351	1.009923	1.018989
		0.4	1.072578	1.043055	1.019804	1.038003
		0.6	1.111484	1.064975	1.029552	1.056882
		0.8	1.151992	1.086946	1.039064	1.075442
	2	0.4	1.05128	1.035127	1.016354	1.03305
		0.8	1.106139	1.070595	1.032115	1.065347
		1.2	1.16347	1.105137	1.046517	1.095442
		1.6	1.221552	1.137098	1.058617	1.121534
	3	0.6	1.055876	1.042417	1.019961	1.042504
		1.2	1.114381	1.083564	1.03784	1.081584
		1.8	1.17102	1.118989	1.05107	1.112097
		2.4	1.219317	1.143075	1.056692	1.127955
	4	0.8	1.054002	1.044472	1.021317	1.047722
		1.6	1.106429	1.083893	1.037977	1.086781
		2.4	1.146846	1.10838	1.044476	1.105459
		3.2	1.163072	1.107772	1.035869	1.092745
10	2	0.4	1.084894	1.04809	1.021914	1.041102
		0.8	1.178396	1.097043	1.043115	1.081547
		1.2	1.279312	1.145224	1.062591	1.119604
		1.6	1.385402	1.190452	1.0791	1.153093
	4	0.8	1.096995	1.069233	1.03134	1.064825
		1.6	1.199101	1.132872	1.056221	1.119378
		2.4	1.290302	1.17631	1.066684	1.148255
		3.2	1.34899	1.183809	1.055469	1.136548
	6	1.2	1.08406	1.069734	1.032178	1.07286
		2.4	1.148883	1.113334	1.045204	1.109544
		3.6	1.154209	1.095547	1.021519	1.071604
		4.8	1.087902	1.013233	0.964502	0.961798
	8	1.6	1.063273	1.057524	1.027502	1.067768
		3.2	1.074836	1.06072	1.020023	1.061273
		4.8	1.000936	0.980541	0.965693	0.948591
		6.4	0.89645	0.874932	0.902132	0.809284
15	3	0.6	1.141573	1.07777	1.034699	1.064567
		1.2	1.303432	1.155534	1.066246	1.125463
		1.8	1.478519	1.225872	1.090328	1.17536
		2.4	1.653338	1.278982	1.101898	1.205339
	6	1.2	1.133404	1.099866	1.04366	1.093126
		2.4	1.252208	1.168139	1.062459	1.143924
		3.6	1.29542	1.155397	1.032775	1.103963
		4.8	1.232934	1.053171	0.958431	0.974106
	9	1.8	1.094306	1.083054	1.036984	1.089664
		3.6	1.096105	1.068807	1.012581	1.055411
		5.4	0.969683	0.933125	0.924394	0.879913
		7.2	0.829947	0.794106	0.844291	0.710812
	12	2.4	1.052432	1.04982	1.022714	1.062981
		4.8	0.967154	0.961732	0.962796	0.937033
		7.2	0.838228	0.832712	0.885729	0.760458
		9.6	0.771467	0.766912	0.849112	0.672388

Table 3: $h'\gamma_1(h'\gamma + h'Z'X_2\beta_2) = 0$,n=10,k=3

$\lambda_1+\lambda_2+\lambda_3$	$\lambda_1+\lambda_2$	λ_1	SR	PSR	MMSE	AMMSE
5	1	0.2	1.021385	0.356921	0.599144	0.336104
		0.4	0.950499	0.369685	0.608819	0.345412
		0.6	0.897334	0.379258	0.616075	0.352392
		0.8	0.855983	0.386704	0.621718	0.357821
	2	0.4	0.750616	0.449975	0.664427	0.40534
		0.8	0.708017	0.465402	0.675194	0.416839
		1.2	0.680909	0.475219	0.682046	0.424156
		1.6	0.662142	0.482016	0.68679	0.429222
	3	0.6	0.669145	0.534689	0.719542	0.471145
		1.2	0.652245	0.549277	0.729103	0.482442
		1.8	0.642588	0.557613	0.734566	0.488898
		2.4	0.636339	0.563007	0.738101	0.493075
	4	0.8	0.669943	0.611357	0.766967	0.534278
		1.6	0.667111	0.623891	0.774809	0.544504
		2.4	0.665611	0.630526	0.77896	0.549918
		3.2	0.664683	0.634634	0.78153	0.553269
10	2	0.4	1.038846	0.310573	0.568659	0.301845
		0.8	0.918277	0.325031	0.5805	0.31222
		1.2	0.841549	0.334232	0.588035	0.318823
		1.6	0.78843	0.340602	0.593251	0.323394
	4	0.8	0.635918	0.455562	0.671225	0.407415
		1.6	0.605029	0.469173	0.68067	0.417378
		2.4	0.588676	0.476379	0.685671	0.422652
		3.2	0.578552	0.48084	0.688766	0.425917
	6	1.2	0.626674	0.583271	0.751264	0.507514
		2.4	0.624963	0.593487	0.757681	0.515551
		3.6	0.624145	0.598373	0.760749	0.519395
		4.8	0.623665	0.601237	0.762548	0.521648
	8	1.6	0.701858	0.692246	0.815829	0.603136
		3.2	0.705959	0.699157	0.819946	0.609198
		4.8	0.707798	0.702254	0.821792	0.611915
		6.4	0.708842	0.704013	0.822839	0.613458
15	3	0.6	0.98692	0.276968	0.547214	0.277912
		1.2	0.848232	0.290273	0.558784	0.287406
		1.8	0.76898	0.297876	0.565396	0.292832
		2.4	0.717699	0.302795	0.569674	0.296342
	6	1.2	0.562073	0.456392	0.674371	0.405971
		2.4	0.545727	0.467238	0.681777	0.413777
		3.6	0.53791	0.472425	0.685319	0.417511
		4.8	0.533327	0.475465	0.687396	0.4197
	9	1.8	0.626564	0.611612	0.769097	0.52816
		3.6	0.629505	0.618597	0.773338	0.533743
		5.4	0.630792	0.621653	0.775193	0.536186
		7.2	0.631514	0.623367	0.776234	0.537556
	12	2.4	0.739685	0.73788	0.842508	0.644616
		4.8	0.743156	0.741856	0.844824	0.648346
		7.2	0.744595	0.743504	0.845785	0.649892
		9.6	0.745382	0.744407	0.84631	0.650738

Table 4: $h'\gamma_1(h'\gamma + h'Z'X_2\beta_2) = -4$,n=10,k=3

$\lambda_1+\lambda_2+\lambda_3$	$\lambda_1+\lambda_2$	λ_1	SR	PSR	MMSE	AMMSE
5	1	0.2	0.968257	0.971546	0.985329	0.973389
		0.4	0.938861	0.944909	0.971488	0.948296
		0.6	0.91164	0.919981	0.958439	0.92465
		0.8	0.886431	0.896654	0.946141	0.902374
	2	0.4	0.950087	0.953933	0.976063	0.95424
		0.8	0.906855	0.913441	0.954723	0.913492
		1.2	0.869393	0.87786	0.935724	0.877253
		1.6	0.83688	0.846565	0.918807	0.845019
	3	0.6	0.941507	0.944707	0.970853	0.941411
		1.2	0.893756	0.898909	0.946251	0.892053
		1.8	0.854693	0.860942	0.925509	0.85051
		2.4	0.822578	0.829337	0.907978	0.815449
	4	0.8	0.939489	0.941721	0.968686	0.933944
		1.6	0.892651	0.896059	0.943613	0.88119
		2.4	0.856215	0.860151	0.923525	0.83902
		3.2	0.827581	0.831655	0.907318	0.805065
10	2	0.4	0.931517	0.938924	0.968498	0.944388
		0.8	0.873015	0.885343	0.940412	0.894934
		1.2	0.823002	0.838346	0.915406	0.851007
		1.6	0.78017	0.79708	0.89314	0.811979
	4	0.8	0.907849	0.913653	0.954962	0.913165
		1.6	0.837886	0.846398	0.918911	0.844035
		2.4	0.78441	0.793786	0.890035	0.78892
		3.2	0.74307	0.752229	0.866745	0.744646
	6	1.2	0.907866	0.910794	0.952101	0.899984
		2.4	0.84432	0.848218	0.917417	0.828129
		3.6	0.799697	0.803647	0.892074	0.77595
		4.8	0.767519	0.771104	0.873165	0.737219
	8	1.6	0.919146	0.920256	0.955581	0.900225
		3.2	0.867841	0.86921	0.92628	0.835017
		4.8	0.834183	0.835492	0.906483	0.791263
		6.4	0.811102	0.81224	0.892581	0.760702
15	3	0.6	0.893757	0.905238	0.9511	0.915132
		1.2	0.810149	0.827264	0.909847	0.843942
		1.8	0.744133	0.763007	0.875083	0.784249
		2.4	0.691709	0.709815	0.845712	0.734043
	6	1.2	0.872227	0.878597	0.936425	0.876394
		2.4	0.786052	0.794157	0.890456	0.788009
		3.6	0.726709	0.734438	0.856905	0.724066
		4.8	0.684667	0.691104	0.831896	0.676752
	9	1.8	0.885316	0.887339	0.938476	0.868077
		3.6	0.816833	0.819135	0.899746	0.786229
		5.4	0.773948	0.775973	0.874522	0.733477
		7.2	0.745531	0.747123	0.85727	0.697693
	12	2.4	0.909917	0.910343	0.94868	0.880055
		4.8	0.861596	0.862042	0.92017	0.814433
		7.2	0.8335	0.833875	0.903199	0.775764
		9.6	0.815717	0.816006	0.892267	0.75104

Table 5: $h'\gamma_1(h'\gamma + h'Z'X_2\beta_2) = -8$,n=10,k=3

$\lambda_1+\lambda_2+\lambda_3$	$\lambda_1+\lambda_2$	λ_1	SR	PSR	MMSE	AMMSE
5	1	0.2	0.983616	0.986523	0.993178	0.98751
		0.4	0.967806	0.973421	0.986514	0.975317
		0.6	0.952552	0.960687	0.980008	0.963418
		0.8	0.937839	0.948317	0.973659	0.951813
	2	0.4	0.974584	0.97787	0.988707	0.978218
		0.8	0.950778	0.956923	0.977925	0.957444
		1.2	0.928499	0.937123	0.967647	0.937664
		1.6	0.907662	0.918423	0.957864	0.918856
	3	0.6	0.970412	0.973087	0.986057	0.971731
		1.2	0.94339	0.948247	0.973034	0.945377
		1.8	0.91876	0.925379	0.960913	0.920887
		2.4	0.896333	0.904358	0.949657	0.898182
	4	0.8	0.969444	0.971288	0.984818	0.967712
		1.6	0.942148	0.945402	0.970939	0.938267
		2.4	0.917842	0.922159	0.958321	0.911556
		3.2	0.896228	0.901329	0.946886	0.887395
10	2	0.4	0.963547	0.970494	0.985096	0.973426
		0.8	0.929693	0.942604	0.970867	0.948107
		1.2	0.898275	0.916274	0.957306	0.924022
		1.6	0.869132	0.891436	0.944399	0.901139
	4	0.8	0.952065	0.957142	0.978099	0.957356
		1.6	0.909792	0.918658	0.958089	0.918557
		2.4	0.872602	0.884229	0.939905	0.883431
		3.2	0.839907	0.853478	0.923433	0.851718
	6	1.2	0.952173	0.954666	0.976098	0.949618
		2.4	0.911652	0.915775	0.955162	0.905755
		3.6	0.877489	0.882624	0.936998	0.867896
		4.8	0.84869	0.854402	0.921295	0.835308
	8	1.6	0.957677	0.958619	0.977286	0.948536
		3.2	0.923042	0.924526	0.95819	0.905583
		4.8	0.894924	0.896693	0.942345	0.87015
		6.4	0.872063	0.873956	0.929223	0.840946
15	3	0.6	0.942184	0.953361	0.976485	0.958747
		1.2	0.890567	0.910504	0.954538	0.920403
		1.8	0.844541	0.871211	0.934126	0.884871
		2.4	0.803523	0.835233	0.915183	0.852012
	6	1.2	0.932266	0.93789	0.968196	0.937477
		2.4	0.875727	0.884909	0.940374	0.883227
		3.6	0.828693	0.839973	0.916261	0.836533
		4.8	0.789527	0.801887	0.895434	0.79644
	9	1.8	0.938726	0.940477	0.96808	0.930832
		3.6	0.890165	0.892821	0.941839	0.874587
		5.4	0.851932	0.854986	0.920567	0.829379
		7.2	0.821688	0.824844	0.903323	0.792988
	12	2.4	0.950902	0.951277	0.972459	0.935037
		4.8	0.913845	0.914377	0.951148	0.885359
		7.2	0.886123	0.886705	0.934912	0.847835
		9.6	0.865153	0.865729	0.922451	0.819227